ABSTRACT

Leader election is one of the fundamental problems in distributed computing. In its implicit version, only the leader must know who is the elected leader. This paper focuses on studying the message complexity of leader election in synchronous distributed networks, in particular, in networks of diameter two. Kutten et al. [JACM 2015] showed a fundamental lower bound of $\Omega(m)$ (where $m$ is the number of edges in the network) on the message complexity of (implicit) leader election that applied also to Monte Carlo randomized algorithms with constant success probability; this lower bound applies for graphs that have diameter at least three. On the other hand, for complete graphs (i.e., diameter 1), Kutten et al. [TCS 2015] established a tight bound of $\Theta(\sqrt{n})$ on the message complexity of randomized leader election ($n$ is the number of nodes in the network). For graphs of diameter two, the complexity was not known.

In this paper, we settle this complexity by showing a tight bound of $\Theta(n)$ on the message complexity of leader election in diameter-two networks. We first give a simple randomized Monte-Carlo leader election algorithm that with high probability (i.e., probability at least $1 - n^{-c}$, for some positive constant $c$) succeeds and uses $O(n \log^3 n)$ messages and runs in $O(1)$ rounds; this algorithm works without knowledge of $n$ (and hence needs no global knowledge). We then show that any algorithm (even Monte Carlo randomized algorithms with large enough constant success probability) needs $\Omega(n)$ messages (even when $n$ is known), regardless of the number of rounds. We also present an $O(n \log n)$ messages deterministic algorithm that takes $O(\log n)$ rounds (but needs knowledge of $n$); we show that this message complexity is tight for deterministic algorithms.

Our results show that leader election can be solved in diameter-two graphs in (essentially) linear (in $n$) message complexity and thus the $\Omega(m)$ lower bound does not apply to diameter-two graphs. Together with the two previous results of Kutten et al., our results fully characterize the message complexity of leader election vis-à-vis the graph diameter.

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1 Notation $\tilde{\Omega}$ hides a polylog(n) factor; $\tilde{O}$ and $\tilde{\Theta}$ hide a polylog(n) factor.

KEYWORDS

Distributed Algorithm, Leader Election, Randomized Algorithm, Message Complexity, Time Complexity, Lower Bounds

1 INTRODUCTION

Leader election is a classical and fundamental problem in distributed computing. The leader election problem requires a group of processors in a distributed network to elect a unique leader among themselves, i.e., exactly one processor must output the decision that it is the leader, say, by changing a special status component of its state to the value leader [13]. All the rest of the nodes must change their status component to the value non-leader. These nodes need not be aware of the identity of the leader. This implicit variant of leader election is quite standard (cf. [13]), and has been extensively studied (see e.g., [10] and the references therein) and is sufficient in many applications, e.g., for token generation in a token ring environment [12]. In this paper, we focus on this implicit variant.2

The complexity of leader election, in particular, its message and time complexity, has been extensively studied both in general graphs as well as in special graph classes such as rings and complete networks, see e.g., [10, 11, 13, 16, 18, 19]. While much of the earlier work focused on deterministic algorithms, recent works have studied randomized algorithms (see e.g., [10, 11] and the references therein). Kutten et al. [10] showed a fundamental lower bound of $\Omega(m)$ (where $m$ is the number of edges in the network) on the message complexity of (implicit) leader election that applied even to Monte Carlo randomized algorithms with (large-enough) constant success probability; this lower bound applies for graphs that have diameter at least three.

We point that the $\Omega(m)$ lower bound applies even for algorithms that have knowledge of $n, m, D$ (throughout, $n$ denotes the number of nodes, $m$ the number of edges, and $D$ the network diameter). The lower bound proof involves constructing a “dumb-bell” graph $G$ which consists of two regular subgraphs $G_1$ and $G_2$ (each having approximately $\frac{n}{2}$ edges) joined by a couple of “bridge” edges (the bridge edges are added so that the regularity is preserved). Note that (even) if $G_1$ and $G_2$ are cliques (in particular, they can be any...
2-connected graph) then $G$ will be of diameter (at least) three. This is the smallest diameter that makes the lower bound proof work; we refer to [10] for details.

On the other hand, for complete graphs (i.e., diameter one), Kutten et al. [11] established a tight bound of $O(\sqrt{n})$ on the message complexity of randomized leader election (for the number of nodes in the network). In other words, they showed an $O(\sqrt{n})$ messages algorithm that elects a (unique) leader with high probability. To complement this, they also showed that any leader election algorithm in a complete graph requires $\Omega(\sqrt{n})$ messages to succeed with (large-enough) constant probability.

For graphs of diameter two, the message complexity was not known. In this paper, we settle this complexity by showing a tight bound of $O(n)$ on the message complexity of leader election in diameter-two networks. In particular, we present a randomized leader election algorithm that takes $O(n \log^3 n)$ messages and $O(1)$ rounds that works even when $n$ is not known. In contrast, we show that any randomized algorithm (even Monte Carlo algorithms with constant success probability) needs $\Omega(n)$ messages. Our results show that leader election can be solved in diameter-two graphs in (essentially) linear (in $n$) message complexity which is optimal (up to a polylog($n$) factor) and thus the $\Omega(m)$ message lower bound does not apply to diameter-two graphs. Together with the previous results [10, 11], our results fully characterize the message complexity of leader election vis-à-vis the graph diameter (see Table 1).

### 1.1 Our Results

This paper focuses on studying the message complexity of leader election (both randomized and deterministic) in synchronous distributed networks, in particular, in networks of diameter two.

For our algorithms, we assume that the communication is synchronous and follows the standard $CONGEST$ model [17], where a node can send in each round at most one message of size $O(\log n)$ bits on a single edge. We assume that the nodes have unique IDs. We assume that all nodes wake up simultaneously at the beginning of the execution. (Additional details on our distributed computation model are given in Section 1.3.)

We show the following results:

1. **Algorithms:** We show that the message complexity of leader election in diameter-two graphs is $O(n)$, by presenting a randomized (implicit) leader election algorithm (cf. Section 2), that takes $O(n \log^3 n)$ messages and runs in $O(1)$ rounds with high probability (whp).

   This algorithm works even without knowledge of $n$. While it is easy to design an $O(n \log n)$ messages randomized algorithm with knowledge of $n$ (Section 1.2), not having knowledge of $n$ makes the analysis more involved. We also present a deterministic algorithm that uses only $O(n \log n)$ messages, but it takes $O(\log n)$ rounds. Also this algorithm needs knowledge (or a constant factor upper bound) of $n$ (or $\log n$) (cf. Section 4). It is not difficult to convert this algorithm (under the same bounds) to solve explicit leader election, where the identity of the leader is broadcast to all nodes. Thus broadcast, another fundamental problem, can be solved in diameter-two graphs in $O(n \log n)$ messages and $O(\log n)$ rounds if $n$ is known (in contrast we note that $\Omega(\log n)$ is a lower bound for broadcast on graphs of diameter at least three, even if $n$ is known and even for randomized algorithms [10]). (We note that all our algorithms will work seamlessly for complete networks as well.)

2. **Lower Bounds:** We show that, in general, it is not possible to improve over our algorithm substantially, by presenting a lower bound for leader election that applies also to randomized (Monte Carlo) algorithms. We show that $\Omega(n)$ messages are needed for any leader election algorithm (regardless of the number of rounds) in a diameter-two network which succeeds with any constant probability that is strictly larger than $\frac{1}{2}$ (cf. Section 3). This lower bound holds even in the $LOCAL$ model [17], where there is no restriction on the number of bits that can be sent on each edge in each round. To the best of our knowledge, this is the first non-trivial lower bound for randomized leader election in diameter-two networks.

We also show a simple deterministic reduction that shows that any super-linear message lower bound for complete networks also applies to diameter-two networks as well (cf. Section 5). It can be shown that $\Omega(n \log n)$ messages is a lower bound for deterministic leader election in complete networks [1, 9] (under assumption that the number of rounds is bounded by some function of $n$) and by our reduction this lower bound also applies for diameter-two networks. (We point out that lower bounds for complete networks do not directly translate to diameter-two networks.)

### Table 1: Message and time complexity of leader election.

<table>
<thead>
<tr>
<th>Diameter</th>
<th>Randomized Time</th>
<th>Messages</th>
<th>Deterministic Time</th>
<th>Messages</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D = 1$: [1, 11]</td>
<td>Upper Bound $O(1)$</td>
<td>$O(\sqrt{n} \log^2 n)$</td>
<td>Lower Bound $\Omega(1)$</td>
<td>$\Omega(\sqrt{n})$</td>
</tr>
<tr>
<td></td>
<td>$O(1)$†</td>
<td>$O(n \log n)$†</td>
<td>$\Omega(1)$</td>
<td>$\Omega(\log n)$</td>
</tr>
<tr>
<td>$D = 2$:</td>
<td>Upper Bound $O(1)$</td>
<td>$O(n \log^3 n)$</td>
<td>Lower Bound $\Omega(1)$</td>
<td>$\Omega(n)$</td>
</tr>
<tr>
<td></td>
<td>$O(n \log n)$†</td>
<td>$O(\log n)$†</td>
<td>$\Omega(1)$</td>
<td>$\Omega(n \log n)$</td>
</tr>
<tr>
<td>$D \geq 3$: [10]</td>
<td>Upper Bound $O(D)$</td>
<td>$O(\log \log n)$</td>
<td>Lower Bound $\Omega(D)$</td>
<td>$\Omega(m)$</td>
</tr>
<tr>
<td></td>
<td>$O(D \log n)$</td>
<td>$O(\log n)$</td>
<td>$\Omega(D)$</td>
<td>$\Omega(m)$</td>
</tr>
</tbody>
</table>

† Note that attaining $O(1)$ time requires $\Omega(n^{1+\Omega(1)})$ messages in cliques, whereas achieving $O(n \log n)$ messages requires $\Omega(\log n)$ rounds; see [1].

§ Needs knowledge of $n$.

†† Note that it is easy to show a $O(1)$ round deterministic algorithm that takes $O(m)$ messages.

### 1.2 Technical Overview

All our algorithms exploit the following simple “neighborhood intersection” property of diameter-two graphs: Any two nodes (that are non-neighbors) have at least one neighbor in common (please refer to Observation 1). However, note that unlike complete networks (which have been extensively studied with respect to leader election — cf. Section 1.4), in diameter-two networks, nodes generally don’t

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1. Throughout, “with high probability” means with probability at least $1 - 1/n^e$, for some constant $e > 0$.

2. Aref and Gafni [1] show the $\Omega(n \log n)$ message lower bound for complete networks under the non-simultaneous wake-up model in synchronous networks. The same message bound can be shown to hold in the simultaneous wake-up model as well under the restriction that the number of rounds is bounded by a function of $n$ [9].
have knowledge of \( n \), the network size (in a complete graph, this is trivially known by the degree). This complicates obtaining sublinear in \( m \) (where \( m \) is the number of edges) message algorithms that are fully localized (don’t have knowledge of \( n \)). Indeed, if \( n \) is known, the following is a simple randomized algorithm: each node becomes a candidate with probability \( \Theta(\frac{\log n}{n}) \) and sends its ID to all its neighbors; any node that gets one or more messages acts as a “referee” and notifies the candidate that has the smallest ID (among those it has received). The neighborhood intersection property implies that at least one candidate will be chosen uniquely as the leader with high probability.

If \( n \) is not known, the above idea does not work. However, we show that if each node \( v \) becomes a candidate with probability \( \frac{1 + \log d(v)}{d(v)} \), (where \( d(v) \) is the degree of \( v \)) then the above idea can be made to work. The main technical difficulty is then showing that at least one candidate is present (cf. Section 2.1) and in bounding the message complexity (cf. Section 2.2). We use Lagrangian optimization to prove that on expectation at least \( \Theta(\log n) \) candidates will be selected and then use a Chernoff bound to show a high probability result.

Our \( \Omega(n) \) randomized lower bound is inspired by the bridge crossing argument of [10] and [15]. In this argument, we construct a “dumbbell” graph \( G \) which is done by taking two identical regular graphs \( G_1 \) and \( G_2 \), removing an edge from each and adding them as bridge edges between \( G_1 \) and \( G_2 \) (so that regularity is preserved). The argument is that any leader election algorithm should send at least one message across one of the two bridge edges (bridge crossing); otherwise, it can be shown that the executions in \( G_1 \) and \( G_2 \) are identical leading to election of two leaders which is not valid. The argument in [10] shows that \( \Omega(m) \) messages are needed for bridge crossing. As pointed out earlier in Section 1, this construction makes the diameter of \( G \) at least three and hence does not work for diameter-two graphs. To overcome this, we modify the construction that takes two complete graphs and add a set of bridge edges (as opposed to just two); see Fig 1. This creates a diameter-two graph; however, the large number of bridge edges requires a different style of argument and results in a bound different compared to [10]. We show that \( \Omega(n) \) messages (in expectation) are needed to send a message across at least one bridge.

We also present a deterministic algorithm that uses \( O(n \log n) \) messages, but takes \( O(\log n) \) rounds. Note that, in a sense, this improves over the randomized algorithm that sends \( O(n \log^3 n) \) messages (although, we did not strive to optimize the log factors). However, the deterministic algorithm is slower by a \( \log(n) \)-factor and is more involved compared to the very simple randomized algorithm (although its analysis is a bit more complicated). Our deterministic algorithm uses ideas similar to Afek and Gafni’s [1] leader election algorithm for complete graphs; however, the algorithm is a bit more involved. Our algorithm assumes knowledge of \( n \) (this is trivially true in complete networks, since every node can infer \( n \) from its degree) which is needed for termination. It is not clear if one can design an \( O(n \log n) \) messages algorithm (running in \( O(\log n) \) rounds) that does not need knowledge of \( n \), which is an interesting open question (cf. Section 6).

Finally, we present a simple reduction that shows that superlinear (in \( n \)) lower bounds in complete networks also imply lower bounds for diameter-two networks, by showing how using only \( O(n) \) messages and in \( O(1) \) rounds, a complete network can be converted to a diameter-two network in a distributed manner. This shows that our deterministic algorithm (cf. Section 4) is message optimal.

1.3 Distributed Computing Model

The model we consider is similar to the models of [1, 3, 5, 7, 8], with the main addition of giving processors access to a private unbiased coin. We consider a system of \( n \) nodes, represented as an undirected graph \( G = (V, E) \). In this paper, we focus on graphs with diameter \( D(G) = 2 \), where \( D(G) \) is the diameter of \( G = (V, E) \). An obvious consequence of this is that \( G \) is connected, therefore \( n - 1 \leq m \leq \frac{n(n-1)}{2} \), where \( m = |E| \) and \( n = |V| \).

Each node has a unique identifier (ID) of \( \Theta(\log n) \) bits and runs an instance of a distributed algorithm. The computation advances in synchronous rounds where, in every round, nodes can send messages, receive messages that were sent in the same round by neighbors in \( G \), and perform some local computation. Every node has access to the outcome of unbiased private coin flips (for randomized algorithms). Messages are the only means of communication; in particular, nodes cannot access the coin flips of other nodes, and do not share any memory. Throughout this paper, we assume that all nodes are awake initially and simultaneously start executing the algorithm. We note that initially nodes have knowledge only of themselves, in other words we assume the clean network model — also called the KT0 model [17] which is standard and most commonly used. On the other hand, if one assumes the KT1 model, where nodes have an initial knowledge of the IDs of their neighbors, there exists a trivial algorithm for leader election in a diameter-two graph that uses only \( O(n) \) messages.

1.4 Other Related Works

The complexity of the leader election problem and algorithms for it, especially deterministic algorithms (guaranteed to always succeed), have been well-studied. Various algorithms and lower bounds are known in different models with synchronous/asynchronous communication and in networks of varying topologies such as a cycle, a complete graph, or some arbitrary topology (e.g., see [4, 10, 11, 13, 16, 18, 19] and the references therein).

The study of leader election algorithms is usually concerned with both message and time complexity. We discuss two sets of results, one for complete graphs and the other for general graphs. As mentioned earlier, for complete graphs, Kutten et al. [11] showed that \( \Theta(\sqrt{n}) \) is the tight message complexity bound for randomized (implicit) leader election. In particular, they presented an \( O(\sqrt{n} \log^{3/2} n) \) messages algorithm that ran in \( O(1) \) rounds; they also showed an almost matching lower bound for randomized leader election, showing that \( \Omega(\sqrt{n}) \) messages are needed for any leader election algorithm that succeeds with a sufficiently large constant probability.

For deterministic algorithms on complete graphs, it is known that \( \Theta(n \log n) \) is a tight bound on the message complexity [1, 9]. In particular, Afek and Gafni [1] presented an \( O(n \log n) \) messages algorithm for complete graphs that ran in \( O(\log n) \) rounds. For complete graphs, Korach et al. [6] and Humblet [3] also presented \( O(n \log n) \) message algorithms. Afek and Gafni [1] presented asynchronous and synchronous algorithms, as well as a tradeoff between the message
and the time complexity of synchronous deterministic algorithms for complete graphs: the results varied from a $O(1)$-time, $O(n^2)$-messages algorithm to a $O(\log n)$-time, $O(n \log n)$-messages algorithm. Afek and Gafni [1], as well as [6, 8] showed a lower bound of $O(n \log n)$ messages for deterministic algorithms in the general case.5

For general graphs, the best known bounds are as follows. Kutten et al. [10] showed that $\Omega(m)$ is a very general lower bound on the number of messages and $\Omega(D)$ is a lower bound on the number of rounds for any leader election algorithm. It is important to point out that their lower bounds applied for graphs with diameter at least three. Note that these lower bounds hold even for randomized Monte Carlo algorithms that succeed even with (some large enough, but) constant success probability and apply even for implicit leader election. Earlier results, showed such lower bounds only for deterministic algorithms and only for the restricted case of comparison algorithms, where it was also required that nodes may not wake up spontaneously and that $D$ and $n$ were not known. The $\Omega(m)$ and $\Omega(D)$ lower bounds are universal in the sense that they hold for all universal algorithms (namely, algorithms that work for all graphs), apply to every $D \geq 3$, $m$, and $n$, and hold even if $D$, $m$, and $n$ are known, all the nodes wake up simultaneously, and the algorithms can make any use of node’s identities. To show that these bounds are tight, they also present an $O(m)$ messages algorithm (this algorithm is not time-optimal). An $O(D)$ time leader election algorithm is known [16] (this algorithm is not message-optimal). They also presented an $O(m \log \log n)$ messages randomized algorithm that ran in $O(D)$ rounds (where $D$ is the network diameter) that is simultaneously almost optimal with respect to both messages and time. They also presented an $O(m \log n)$ and $O(D \log n)$ deterministic leader election algorithm for general graphs.

2 A RANDOMIZED ALGORITHM

In this section, we present a simple randomized Monte Carlo algorithm that works in a constant number of rounds. Algorithm 1 is entirely local, as nodes do not require any knowledge of $n$. Nevertheless, we show that we can sub-sample a small number of candidates (using only local knowledge) that then attempt to become leader. In the remainder of this section, we prove the following result.

THEOREM 2.1. There exists a Monte Carlo randomized leader election algorithm that, with high probability, succeeds in $n$-node networks of diameter at most two in $O(1)$ rounds, while sending $O(n \log^3 n)$ messages.

2.1 Proof of Correctness: Analyzing the number of candidates selected

We use the following property of diameter-2 graphs crucially in our algorithm.

Algorithm 1 Randomized leader election in $O(1)$ rounds and $O(n \log^3 n)$ message complexity

1. Each node $v \in V$ selects itself to be a “candidate” with probability $1 + \log d_v$, where $d_v$ is the degree of $v$.
2. If $v$ becomes a candidate then $v$ sends its ID to all its neighbors.
3. Each node acts as a “referee node” for all its candidate neighbors (including, possibly itself).
4. If a node $w$ receives ID’s from its neighbors $v_1, v_2, \ldots, v_j$ (say), then $w$ computes the minimum ID of those and sends it back to those neighbors. That is, $w$ sends $\min \{ID(v_1), ID(v_2), \ldots, ID(v_j)\}$ back to each of $v_1, v_2, \ldots, v_j$.
5. A node $v$ decides that it is the leader if and only if it receives its own ID from all its neighbors. Otherwise $v$ decides that it is not the leader.

Observation 1. Let $G = (V, E)$ be a graph of diameter 2. Then for any $u, v \in V$, either $(u, v) \in E$ or $3w \in V$ such that $(u, w) \in E$ and $(v, w) \in E$, i.e., $u$ and $v$ has at least one common neighbor $w$ (say).

We note that if one or more candidates are selected, then only the candidate node with the minimum ID is selected as the leader. That is, the leader is unique, and therefore the algorithm produces the correct output. The only case when the algorithm may be wrong is if no candidates are selected to begin with, in which case no leader is selected. In this section, we show that, with high probability, at least two candidates are

LEMMA 2.2. Let $f(x_1, x_2, \ldots, x_n)$ be a function of $n$ variables $x_1, x_2, \ldots, x_n$, where $x_1, x_2, \ldots, x_n$ are positive reals. $f$ is defined as

$$f(x_1, x_2, \ldots, x_n) \triangleq \sum_{i=1}^{n} \frac{1 + \log x_i}{x_i}$$

Let $C$ be a constant $\geq n \sqrt{2}$. Then $f(x_1, x_2, \ldots, x_n)$ is minimized, subject to the constraint $\sum_{i=1}^{n} x_i = C$, when $x_i = \frac{C}{n}$, for all $1 \leq i \leq n$. The minimum value that $f(x_1, x_2, \ldots, x_n)$ takes is at the point $(\frac{C}{n}, \frac{C}{n}, \ldots, \frac{C}{n})$, and is given by

$$f_{\min} = f\left(\frac{C}{n}, \frac{C}{n}, \ldots, \frac{C}{n}\right) = \frac{n^2}{C} (1 + \log \left(\frac{C}{n}\right)).$$

PROOF. We use standard Lagrangian optimization techniques to show this. Please refer to the appendix for the full proof.

LEMMA 2.3. Let $X$ be a random variable that denotes the total number of candidates selected in Algorithm 1. Then the expected number of selected candidates is lower-bounded by $E[X] > 2 + \frac{1}{2} \log n$.

PROOF. Let $X_u$ be an indicator random variable that takes the value 1 if and only if $u$ becomes a candidate. Then $E[X_u] = \Pr[X_u = 1] = \frac{1 + \log d_u}{d_u}$. Thus if $X$ denotes the total number of candidates selected, then

$$E[X] = \sum_{v \in V} E[X_v] = \sum_{v \in V} \frac{1 + \log (d_v)}{d_v}.$$
Since $G$ is connected, $m \geq n - 1 \implies 2m \geq 2n - 2 > n\sqrt{2}$.
Thus by By Lemma 2.2, $E[X]$ is minimized subject to the constraint
\[ \sum_{v \in V(G)} d_v = 2m \text{ when } d_v = \frac{2m}{n}, \text{ i.e., when } G \text{ is regular.} \]

**Case 1** ($n - 1 \leq m \leq n^2$): The minimum value that $E[X]$ takes is given by
\[
E[X]_{\text{min}} = \frac{n^2}{2m}(1 + \log \left(\frac{2m}{n}\right)) > \frac{n^2}{2m} \quad \text{(since } 1 + \log \left(\frac{2m}{n}\right) > 1) \]
\[
= 1 + \log 2 + \log \left(\frac{n^2}{2}\right) = 2 + \log n. \quad \square
\]

We use the following variant of Chernoff Bound [14] to show concentration, i.e., to show that the number of candidates selected is not too less than its expected value.

**Theorem 2.4 (Chernoff Bound).** Let $X_1, X_2, \ldots, X_n$ be independent indicator random variables, and let $X = \sum_{i=1}^{n} X_i$. Then the following Chernoff bound holds: for $0 < \delta < 1$,
\[
Pr[X \leq (1 - \delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)}\right)^\mu, \quad \text{where } \mu \overset{\text{def}}{=} E[X].
\]

**Lemma 2.5.** If $X$ denotes the number of candidates selected, then $Pr[X \leq 1] < n^{-3}$.

**Proof.** We use Theorem 2.4 to show this. Please refer to the appendix for the full proof. \(\square\)

### 2.2 Computing the message complexity

Note that the expected total message complexity of the algorithm can be bounded as follows. Let random variable $M^{\text{entire}}$ denote the total messages sent during the course of the algorithm. Let $M_v$ be the number of messages sent by node $v$. Thus $M^{\text{entire}} = \sum_{v \in V} M_v$.

A node $v$ becomes a candidate with probability $\frac{1 + \log d_v}{d_v}$, and, if it does, it sends $d_v$ messages (the referees reply to these, but this increases the total number of messages by a factor of 2). Hence by linearity of expectation, it follows that $E[M^{\text{entire}}] = \sum_{v \in V} E[M_v] = \sum_{v \in V} 2 \frac{1 + \log d_v}{d_v} = \sum_{v \in V} (1 + \log d_v) \leq 2 \sum_{v \in V} (1 + \log n) \leq 2n + 2n \log n$. To show concentration, we cannot directly apply a standard Chernoff bound that works for 0-1 random variables, since $M_v$s are not 0-1 random variables (they take values either 0 or $d_v$).

To handle this, we bucket the degrees into (at most) $\log n$ categories based on their value then use a Chernoff bound as detailed below.

We use the following variant of Chernoff Bound [14] in the following analysis.

**Theorem 2.6 (Chernoff Bound).** Let $X_1, X_2, \ldots, X_n$ be independent indicator random variables, and let $X = \sum_{i=1}^{n} X_i$. Then the following Chernoff bound holds: for $R \geq 6E[X]$, $Pr[X \geq R] \leq 2^{-R}$.

**Definition 2.7.** Let $k$ be a positive integer such that $2^{k-1} < n \leq 2^{k}$. For $0 \leq j \leq k$, let $V_j \subset V$ be the set of vertices with degree in $(2^{j-1}, 2^j]$, i.e., if $v \in V_j$, then $2^{j-1} < d_v \leq 2^j$.

**Remark 1.** We note that $\sum_{j=0}^{k} n_j = n$, where $n_j = |V_j|$ for $0 \leq j \leq k$. In particular, $n_j \leq n$ for all $j \in [0, k]$.

**Analyzing vertices with degree $\leq 2$:** We recall that $X_v$ is an indicator random variable that takes the value 1 if and only if $v$ becomes a candidate. Then $Pr[X_v = 1] = 1$ if $v \in V_0 \cup V_1$, i.e., every vertex with degree 1 or degree 2 selects itself to be a candidate, deterministically.

For $v \in V$, let $m_v$ denote the number of messages that $v$ sends. So $m_v = d_v$ if $v$ becomes a candidate, and $m_v = 0$ otherwise. Let $M_j$ be the total number of messages that members of $V_j$ send, i.e.,
\[
M_j \overset{\text{def}}{=} \sum_{v \in V_j} m_v \leq \sum_{v \in V_j} d_v \leq 2^j = n_j; 2^j \leq n; 2^j.
\]

**Analyzing vertices with degree $> 2$:** We recall that for $v \in V$, $X_v$ is an indicator random variable that takes the value 1 if and only if $v$ becomes a candidate. Let $i$ be an integer in $[2, k]$ and let $v \in V_i$.

**Observation 2.** \(\frac{i}{2^i} < E[X_v] < \frac{3i}{2^i}\).

**Proof.** For $v \in V_i$, \(2^{i-1} < d_v \leq 2^i\). So
\[
E[X_v] = Pr[X_v = 1] \quad \text{(since } X_v \text{ is an indicator random variable)}
\]
\[
= \frac{1 + \log d_v}{d_v}
\]
\[
= \frac{1 + \log (2^{i-1})}{2^i} \leq E[X_v] \leq \frac{1 + \log (2^i)}{2^i}
\]
\[\text{(since } 2^{i-1} < d_v \leq 2^i)\]
\[\text{or, } \frac{i}{2^i} < E[X_v] < \frac{i + 1}{2^{i+1}} \leq \frac{3i}{2^i} \]
\[\text{[since } i \geq 2 \implies \frac{3i}{2} \geq i + 1] \quad \square
\]

For $0 \leq j \leq k$, let $Y_j$ be a random variable that denotes the total number of candidates selected from $V_j$.

**Observation 3.** For $2 \leq i \leq k$, \(\frac{i}{2^i} < E[Y_i] < \frac{3i}{2^i}\).

**Proof.**
\[
Y_i = \sum_{v \in V_i} X_v \implies E[Y_i] = E[\sum_{v \in V_i} X_v]
\]
\[
= \sum_{v \in V_i} E[X_v] \quad \text{[by linearity of expectation]}
\]
\[
\implies \sum_{v \in V_i} \frac{i}{2^i} < E[Y_i] < \frac{3i}{2^i}
\]
\[\implies \frac{i}{2^i} < E[Y_i] < \frac{3i}{2^i} \quad \square
\]
Remark 2. \( \forall u, v \in V(G), u \neq v, X_u \) and \( X_v \) are independent, and for \( 0 \leq j \leq k \), we define \( Y_j \) as \( Y_j = \sum_{v \in Y_j} X_v \), i.e., \( Y_j \) is a sum of independent indicator random variables. Hence we can use Theorem 2.6 to show that \( Y_j \) is concentrated around its mean, i.e., its expectation.

We recall that for \( 0 \leq j \leq k \), \( M_j \) is the total number of messages that members of \( V_j \) send, i.e., for \( 2 \leq i \leq k \),

\[
M_i = \sum_{v \in V_i} m_v = \sum_{v \in V_i, X_v = 1} d_v
\]

Lemma 2.8. For any integer \( i \in [2, k] \), it holds that \( Pr[M_i \geq 24n \log^2 n] \leq \frac{1}{n^3} \).

Proof.

\[
M_i = \sum_{v \in V_i} m_v = \sum_{v \in V_i, X_v = 1} d_v
\]

\[
\Rightarrow \sum_{v \in V_i, X_v = 1} 2^{-i+1} < M_i \leq \sum_{v \in V_i, X_v = 1} 2^i \text{ [since } 2^{i-1} < d_v \leq 2^i]\]

\[
\Rightarrow 2^{i-1} \cdot Y_i < M_i \leq 2^i \cdot Y_i
\]

Case 1 (\( E[Y_i] = 0 \)): \( E[Y_i] = 0 \) if and only if \( n_i = 0 \), i.e., if and only if \( \exists v \in V \) such that \( 2^{i-1} < d_v \leq 2^i \). But \( n_i = 0 \Rightarrow V_i = \phi \), the empty set. Therefore, \( M_i = 0 \).

Case 2 (\( 0 < E[Y_i] < 1 \)): Assuming \( n \geq 3, 4 \log n > 6 > 6E[Y_i] \). Therefore, by Theorem 2.6,

\[
Pr[Y_i \geq 4 \log n] \leq 2^{-4\log n} = n^{-4}
\]

\[
\Rightarrow Pr[M_i \geq 2^i \cdot 4 \log n] \leq n^{-4} \text{ [since } M_i \leq 2^i \cdot Y_i]\]

\[
\Rightarrow Pr[M_i \geq 8n \log n] \leq n^{-4} \text{ [since } i < k \text{ < log n + 1]}\]

Case 3 (\( E[Y_i] \geq 1 \)): We have shown before that \( E[Y_i] \leq \frac{3\log n}{27} \). But \( n_i \leq n \text{ for all } 2 \leq i \leq k \). Hence \( E[Y_i] \leq \frac{3n}{27} \). Assuming \( n \geq 3, 4 \log n > 6 \). Therefore, by Theorem 2.6,

\[
Pr[Y_i \geq 12n \log n, \frac{1}{27}] \leq Pr[Y_i \geq 4 \log n E[Y_i]]
\]

\[
\leq 2^{-4 \log n E[Y_i]} = n^{-4E[Y_i]} \leq n^{-4} \text{ [since } E[Y_i] \geq 1]\]

\[
\Rightarrow Pr[M_i \geq 12n \log n] \leq n^{-4} \text{ [since } M_i \leq 2^i \cdot Y_i]\]

\[
\Rightarrow Pr[M_i \geq 24n \log^2 n] \leq Pr[M_i \geq 12n \log n] \leq n^{-4} \text{ [since } i \leq k < \log n + 1 < 2 \log n]\]

\[
\square
\]

Lemma 2.9. If \( M \) denotes the total number of messages sent by the candidates (in the first round only), then \( Pr[M \geq 27n \log^3 n] < \frac{1}{n^3} \).

Proof.

\[
M \overset{\text{def}}{=} \sum_{i=0}^{k} M_i = M_0 + M_1 + \sum_{i=2}^{k} M_i
\]

\[
= n + 2n + \sum_{i=2}^{k} M_i \quad [\text{since } M_0 = n \text{ and } M_1 = 2n]\]

\[
= 3n + \sum_{i=2}^{k} M_i
\]

But for \( 2 \leq i \leq k \), \( Pr[M_i \geq 24n \log^2 n] \leq \frac{1}{n^3} \). Taking the union bound over \( 2 \leq i \leq k \),

\[
Pr[M_i \geq 24n \log^2 n] \text{ for some } i' \in [2, k] \text{ is } \leq \frac{\log n}{n^3} < \frac{1}{n^3}
\]

\[
\Rightarrow Pr[\sum_{i=2}^{k} M_i \geq 24n \log^3 n] < \frac{1}{n^3}
\]

\[
\Rightarrow Pr[3n + \sum_{i=2}^{k} M_i \geq 3n + 24n \log^3 n] < \frac{1}{n^3}
\]

\[
\Rightarrow Pr[M \geq 3n + 24n \log^3 n] < \frac{1}{n^3} \quad [\text{since } M = 3n + \sum_{i=2}^{k} M_i]
\]

But \( 3n \leq 3n \log^3 n \text{ for } n \geq 2, \text{ or, } 3n + 24n \log^3 n \leq 27n \log^3 n \). Hence

\[
Pr[M \geq 27n \log^3 n] \leq Pr[M \geq 3n + 24n \log^3 n] < \frac{1}{n^3}
\]

\[
\square
\]

Lemma 2.10. If \( M^{\text{entire}} \) denotes the total number of messages sent during the entire run of Algorithm 1, then \( Pr[M^{\text{entire}} \geq 54n \log^3 n] < \frac{1}{n^3} \).

Proof. Let \( M' \) denote the number of messages sent by the “referee” nodes in the second round of the algorithm. We recall that \( M \) is the number of messages sent by the “candidate” nodes in the first round of the algorithm. Then \( M' \leq M \), and \( M^{\text{entire}} = M + M' \leq 2M \), and the result follows.

\[
\square
\]

This completes the proof of Theorem 2.1.

3 A LOWER BOUND FOR RANDOMIZED ALGORITHMS

In this section we show that \( \Omega(n) \) is a lower bound on the message complexity for solving leader election with any randomized algorithm in diameter-two networks. Notice that [11] show a lower bound of \( \Omega(\sqrt{\delta}) \) for the special case of diameter 1 networks, and we know from [10] that, for the message complexity becomes \( \Omega(n) \) for (most) diameter 3 networks. Thus, Theorem 3.1 completes the picture regarding the message complexity of leader election when considering networks according to their diameter.

Theorem 3.1. Any algorithm that solves implicit leader election with probability at least \( \frac{1}{2} + \varepsilon \) in any \( n \)-node network with diameter at most 2, for any constant \( \varepsilon > 0 \), sends at least \( \Omega(n) \) messages in expectation. This holds even if nodes have unique IDs and know the network size \( n \).
In the remainder of this section, we prove Theorem 3.1. Assume towards a contradiction, that there is an algorithm that elects a leader with probability $\frac{1}{2} + \varepsilon$ that sends $o(n)$ messages with probability approaching 1. In other words, we assume that the event where the algorithm sends more than $o(n)$ messages (of arbitrary size) happens with probability at most $o(1)$.

**Unique IDs vs Anonymous.** Before describing our lower bound construction, we briefly recall a simple reduction used in [11] that shows that assuming unique IDs does not change the success probability of the algorithm by more than $\frac{1}{n}$. Since we assume that nodes have knowledge of $n$, it is straightforward to see that nodes can obtain unique IDs (whp) by choosing a random integer in the range $[1, n^c]$, for some constant $c \geq 4$. Thus, we can simulate an algorithm that requires unique IDs in the anonymous case and the simulation will be correct with high probability. Suppose that there can obtain unique IDs (whp) by choosing a random integer in the range $[1, n^c]$, for some constant $c \geq 4$. Thus, we can simulate an algorithm that requires unique IDs in the anonymous case and the simulation will be correct with high probability. Suppose that there is an algorithm $A$ that can break the message complexity bound of Theorem 3.1 while succeeding with probability $\geq \frac{1}{2} + \varepsilon$, for some constant $\varepsilon > 0$, when nodes have unique IDs. Then, the above simulation yields an algorithm $A'$ that works in the case where nodes are anonymous with the same message complexity bound as algorithm $A$ and succeeds with probability at least $\left(\frac{1}{2} + \varepsilon - \frac{1}{n}\right) \geq \frac{1}{2} + \varepsilon'$, for some constant $\varepsilon' > 0$. We conclude that proving the lower bound for the anonymous case is sufficient to imply a lower bound for the case where nodes have unique IDs.

**The Lower Bound Graph.** Our lower bound is inspired by the bridge crossing argument of [10] and [15]. For simplicity, we assume that $\frac{n}{2}$ is an integer. Consider two cliques $C_1$ and $C_2$ of $\frac{n}{2}$ nodes each and let $G'$ be the $n$-node graph consisting of the two (disjoint) cliques. The port numbering of an edge $e = (u, v) \in E(G')$ refers to the port number at $u$ and the respective port number at $v$ that connects $e$. The port numberings of the edges defines an instance of $G'$.

Given an instance of $G'$, we will now describe how to obtain an instance of graph $G$ that has the same node set as $G'$. Fix some arbitrary enumeration $u_1, \ldots, u_n$ of the nodes $6$ in $C_1$ and similarly let $v_1, \ldots, v_n$ be an enumeration of the nodes in $C_2$. To define the edges of $G$, we randomly choose a maximal matching $M_1$ of $\frac{n}{2}$ edges in the subgraph $C_1$. Consider the set of edges $M_2' = \{(u_i, v_j) | \exists (u_i, u_j) \in M_1\}$, which is simply the matching in $C_2$ corresponding to $M_1$ in $C_1$. We define $M_2$ to be a randomly chosen maximal matching on $C_2$ when using only edges in $E(G') \setminus M_2'$. Then, we remove all edges in $M_1 \cup M_2$ from $G'$. So far, we have obtained a graph where each node has one unvired port.

The edge set of $G$ consists of all the remaining edges of $G'$ in addition to the set $M = \{(u_1, v_1), \ldots, (u_{\frac{n}{2}}, v_{\frac{n}{2}})\}$, where we connect these bridge edges by using the unwired ports that we obtained by removing the edges as described above. We say that an edge is an intra-clique edge if it has both endpoints in either $C_1$ or $C_2$. Observe that the intra-clique edges of $G$ are a subset of the intra-clique edges of $G'$. Figure 1 gives an illustration of this construction.

**Lemma 3.2.** Graph $G$ is an $n$-node network of diameter 2 and the port numbering of each intra-clique edge in $G$ is the same as of the corresponding edge in $G'$.

**Proof.** By construction, each node in $C_1$ has the same port numbering in both graphs, except for its (single) incident edge that was replaced with a bridge edge to some node in $C_2$, thus we focus on showing that $G$ has diameter $2$.

We will show that node $u_i \in C_1$ has a path of length $2$ to every other node. Observe that any two nodes $u_i, u_j \in C_1$ both have $\frac{n}{2} - 2$ incident intra-clique edges and since $\frac{n}{2} - 2 > \frac{\sqrt{n}}{10}$ they must both have a common neighbor. Now, consider some node $v_j \in C_2$ and assume that $j \neq i$, as otherwise there is the matching edge $(u_i, v_i) \in M$. If $(u_i, u_j) \in E(G)$, then again the result follows because $(u_i, v_j) \in M$. Otherwise, there must be the path $u_i \rightarrow v_i \rightarrow v_j$, since, by construction, the edge $(v_i, v_j) \in M_2'$ and hence $(u_i, v_j) \notin M_2$. A symmetric argument shows that every node has distance $\leq 2$ from a given node in $C_2$.

A state $\sigma$ of the nodes in $C_1$ is a $\frac{n}{2}$-size vector of the local states of the $n/2$ nodes in $C_1$. Since we assume that nodes are anonymous, a state $\sigma$ that is reached by the nodes in $C_1$, can also be reached by the nodes in $C_2$. More formally, when executing the algorithm on the disconnected network $G'$, we can observe that every possible state $\sigma$ (of $\frac{n}{2}$ nodes) has the same probability of occurring in $C_1$ as in $C_2$. Thus, a state where there is exactly one leader among the $\frac{n}{2}$ nodes of a clique in $G'$, is reached with some specific probability $q$ depending on the algorithm. By a slight abuse of notation, we also use $G'$ and $G$ to denote the event that the algorithm executes on $G'$ respectively $G$. For the probability of the event One, which occurs when there is exactly 1 leader among the $n$ nodes, we get

$$\Pr \left[ \text{One} \mid G' \right] = 2q(1 - q) \leq \frac{1}{2},$$

which holds for any value of $q$. Since $G'$ is disconnected, the algorithm does not need to succeed with nonzero probability when being executed on $G'$. However, below we will use this observation to obtain an upper bound on the probability of obtaining (exactly) one leader in $G$. 

![Figure 1: The lower bound graph construction used in Theorem 3.1](image-url)
Now consider the execution on the diameter 2 network $G$ (obtained by modifying the ports of $G'$ as described above) and let $C_1 \leftrightarrow C_2$ be the event that no message is sent across the bridges between $C_1$ and $C_2$. Since we assume the port numbering model where nodes are unaware of their neighbors initially, it follows by Lemma 3.2 that

$$Pr[One | C_1 \leftrightarrow C_2, G] = Pr[One | G'], \quad (2)$$

Let $M$ be the event that the algorithm sends $o(n)$ messages. Recall that we assume towards a contradiction that $Pr[M | G] = 1 - o(1)$.

**Lemma 3.3.** $Pr[C_1 \leftrightarrow C_2 | G, M] = o(1)$.

**Proof.** The proof is inspired by the guessing game approach of [2] and Lemma 16 in [15]. Initially, any node $u \in C_1$ has $\frac{d}{2} - 1$ ports that are all equally likely (i.e., a probability $p = \frac{1}{\frac{d}{2} - 1}$) to be connected to the (single) bridge edge incident to $u$. As $u$ sends messages to other nodes, it might learn about some of its ports connecting to non-bridge edges and hence this probability can increase over time. However, we condition on event $M$, i.e., the algorithm sends at most $o(n)$ messages in total and hence at least $\frac{d}{2}$ ports of each node $u$ remain unused at any point.

It follows that the probability of some node $u$ to activate a (previously unused) port that connects a bridge edge is at most $\frac{1}{d}$ at ANY point of the execution. Let $X$ be the total number of ports connecting bridge edges that are activated during the run of the algorithm and let $X_u$ be the indicator random variable that is 1 iff node $u$ sends a message across its bridge edge. Let $S_u$ be the number of messages sent by node $u$. It follows by the hypergeometric distribution that

$$E[X_u | G, M] = S_u \frac{1}{\Theta(n)},$$

for each node $u$ and hence,

$$E[X | G, M] = \sum_{u \in V(G)} \frac{S_u}{\Theta(n)} = \frac{1}{\Theta(n)} \sum_{u \in V(G)} S_u = o(1)$$

where we have used the fact that $\sum_{u \in V(G)} S_u = o(n)$ due to conditioning on event $M$. By Markov’s Inequality, it follows that the event $C_1 \leftrightarrow C_2$, i.e., $X \geq 1$, occurs with probability at most $o(1)$. 

We now combine the above observations to obtain

$$Pr[One | G, M] = Pr[One | C_1 \leftrightarrow C_2, G, M] Pr[C_1 \leftrightarrow C_2 | G, M] + Pr[One | C_1 \leftrightarrow C_2, G, M] Pr[C_1 \leftrightarrow C_2 | G, M]$$

$$\leq Pr[One | C_1 \leftrightarrow C_2, G, M] + o(1) \quad \text{(by Lem. 3.3)}$$

$$\leq \frac{1}{2} + o(1), \quad (3)$$

where the last inequality follows by first using (2) and noting that the upper bound (1) still holds when conditioning on the event $M$. Finally, we recall that the algorithm succeeds with probability at least $\frac{1}{2} + \epsilon$ and $Pr[M | G] \geq 1 - o(1)$, which yields

$$\frac{1}{2} + \epsilon \leq Pr[One | G] \leq Pr[One | G, M] + o(1) \leq \frac{1}{2} + o(1),$$

which is a contradiction, since we have assumed that $\epsilon > 0$ is a constant.

## 4 A DETERMINISTIC ALGORITHM

Our algorithm (Algorithm 2) is inspired by the solution of Afek and Gafni [1] for the $n$-node clique. However, there are some complications that we explain below, since we cannot rely on all nodes to be connected by an edge. Note that our algorithm assumes that $n$ (or a constant factor upper bound for $\log n$) is known to all nodes.

For any node $v \in V$, we denote the degree of $v$ by $d_v$, and the ID of $v$ by $ID_v$. At any time-point in the algorithm, $L_v$ denotes the highest ID that $v$ has so far learned (among all the probe messages it has received, in the current round or in some previous round).

The algorithm proceeds as a sequence of $\Theta(\log n)$ phases. Initially every node is a “candidate” and is “active”. Each node $v$ numbers its neighbors from 1 to $d_v$, denoted by $w_{v,1}, w_{v,2}, \ldots, w_{v,d_v}$ respectively. In phase $i$, if a node $v$ is active, $v$ sends probe-messages containing its $ID$ to its neighbors $w_{v,2}, \ldots, w_{v,k}$, where $k = \min \{d_v, 2^i - 1\}$. Each one of them replies back with the highest ID it has seen so far. If any on those $ID$’s is higher than $ID_v$, then $v$ stops being a candidate and becomes inactive. Node $v$ also becomes inactive if it has finished sending probe-messages to all its neighbors. After finishing the $\Theta(\log n)$ phases $v$ becomes leader if it is still a candidate.

The idea behind the algorithm is to exploit the neighborhood intersection property (cf. Observation 1) of diameter-2 networks. Since for any $u, v \in V$, there is an $x \in V'$ that is connected to both $u$ and $v$ (unless $u$ and $v$ are directly connected via an edge) and acts as a “referee” node for candidates $u$ and $v$. This means that $x$ serves to inform $u$ and $v$ who among them is the winner, i.e., has the higher ID. Thus at the end of the algorithm, every node except the one with the highest ID should know that he is not a leader. We present the formal analysis of Theorem 4.1 in Sections 4.1 and 4.2.

**Theorem 4.1.** There exists a deterministic leader election algorithm for $n$-node networks with diameter at most 2 that sends $O(n \log n)$ messages and terminates in $O(\log n)$ rounds.

In the pseudocode and the subsequent analysis we use $v$ and $ID_v$ interchangeably to denote the node $v$.

### 4.1 Proof of Correctness

Define $v^{\max}$ to be the node with the highest ID in $G$.

**Lemma 4.2.** $v^{\max}$ becomes a leader.

**Proof.** Since $v^{\max}$ has the highest ID in $G$, the if-clause of Line 15 of Algorithm 2 is never satisfied for $v^{\max}$. Therefore $v^{\max}$ never becomes a non-candidate, and hence becomes a leader at the end of the algorithm.

**Lemma 4.3.** No other node except $v^{\max}$ becomes a leader.

**Proof.** Consider any $u \in V$ such that $u \neq v^{\max}$.

1. **Case 1 ($v^{\max}$ and $u$ are connected via an edge):** Since $v^{\max}$ has the highest ID in $G$, the if-clause of Line 15 of Algorithm 2 is never satisfied for $v^{\max}$. Therefore $v^{\max}$ becomes inactive only if it has already sent probe-messages to all its neighbors (or $v^{\max}$ never becomes inactive). In particular, $u$ always receives a probe-message from $v^{\max}$ containing $ID_{v^{\max}}$. Since $ID_{v^{\max}} > ID_u$, $u$ becomes a non-candidate at that point (if $u$
Algorithm 2 Deterministic Leader Election in $O(\log n)$ rounds and with $O(n \log n)$ messages: Code for a node $v$

1: $v$ becomes a “candidate” and “active”.
2: $L_v \leftarrow ID_v$.
3: $N_v \leftarrow ID_v$.
4: $v$ numbers its neighbors from 1 to $d_v$, which are called $w_{v,1}, w_{v,2}, \ldots, w_{v,d_v}$ respectively.
5: for phase $i = 1$ to $\Theta(\log n)$ do
6: if $v$ is active then
7: $v$ sends a “probe” message containing its ID to its neighbors $\{w_{v,2^i-1}, \ldots, w_{v,\min\{d_v, 2^i-1\}}\}$.
8: if $d_v \leq 2^i - 1$ then
9: $v$ becomes inactive.
10: end if
11: end if
12: $v$ tells every member of $L_v$ such that both $v$ and $x$ remember their respective probe-messages to $v$.
13: end for
14: if $ID_u > L_v$ then
15: $v$ sends $ID_u$ to $N_v$.
16: $L_v \leftarrow ID_u$.
17: $N_v \leftarrow x$. \quad $v$ remembers neighbor who told $v$ about $L_v$.
18: $v$ becomes “inactive” and “non-candidate”.
19: end if
20: $v$ tells every member of $X$ about $L_v$, i.e., the highest ID it has seen so far.
21: if $v$ is still a candidate then
22: $v$ elects itself to be the leader.
23: end if
24: end if

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was still a candidate until that point) and therefore does not become a leader.

- Case 2 ($v_{\text{max}}$ and $u$ do not have an edge between them): By Observation 1, there is some $x \in V$ such that both $v_{\text{max}}$ and $u$ have edges to $x$. And we have already established that $v_{\text{max}}$ will always send a probe-message to $x$ at some point of time or another.

- Case 2(a) ($u$ does not send a probe-message to $x$): This implies that $u$ became inactive before it could send a probe-message to $x$. But then $u$ could have become inactive only if the if-clause of Line 15 of Algorithm 2 got satisfied at some point. Then $u$ became a non-candidate too at the same time and therefore would not become a leader.

- Case 2(b) ($u$ sends a probe-message to $x$ before $v_{\text{max}}$ does): Suppose $u$ sends a probe-message to $x$ at round $i$ and $v_{\text{max}}$ sends a probe-message to $x$ at round $i'$, where $1 \leq i < i' \leq \log n$. If $x$ had seen an ID higher than $ID_u$ up until round $i$, then $x$ immediately informs $u$ and $u$ becomes a non-candidate.

So suppose not. Then, after round $i$, $x$ sets its local variables $L_x$ and $N_x$ to $ID_u$ and $u$ respectively. Let $j > i$ be the smallest integer such that $x$ receives a probe-message from a neighbor $u'$ at round $j$, where $ID_{u'} > ID_u$. Note that $v_{\text{max}}$ will always send a probe-message to $x$, therefore such a $u'$ exists. Then, after round $j$, $x$ sets its local variables $L_x$ and $N_x$ to $ID_{u'}$ and $u'$ respectively, and informs $u$ of this change. $u$ becomes a non-candidate at that point of time.

- Case 2(c) ($u$ and $v_{\text{max}}$ each sends a probe-message to $x$ at the same time): Since $ID_{v_{\text{max}}}$ is the highest ID in the network, $L_x$ is assigned the value $ID_{v_{\text{max}}}$ at this point, and $x$ tells $u$ about $L_x = ID_{v_{\text{max}}} > ID_u$, causing $u$ to become a non-candidate.

- Case 2(d) ($u$ sends a probe-message to $x$ after $v_{\text{max}}$ does): Suppose $v_{\text{max}}$ sends a probe-message to $x$ at round $i$ and $u$ sends a probe-message to $x$ at round $i'$, where $1 \leq i < i' \leq \log n$. Then $x$ sets its local variables $L_x$ and $N_x$ to $ID_{v_{\text{max}}}$ and $ID_u$, respectively, after round $i$. So when $u$ comes probing at round $i' > i$, $x$ tells $u$ about $L_x = ID_{v_{\text{max}}} > ID_u$, causing $u$ to become a non-candidate.

\[\Box\]

4.2 Message Complexity

**Lemma 4.4.** At the end of round $i$, there are at most $\frac{n}{2^i}$ “active” nodes.

**Proof.** Consider a node $v$ that is active at the end of round $i$. This implies that the if-clause of Line 15 of Algorithm 2 has not so far been satisfied for $v$, which in turn implies that $ID_u > ID_w$, for $1 \leq j \leq 2^i - 1$, therefore none of $w_{v,1}, w_{v,2}, \ldots, w_{v,2^i-1}$ is active after round $i$. Thus for every active node at the end of round $i$, there are at least $2^i - 1$ inactive nodes. We call this set of inactive nodes, together with $v$ itself, the “kingdom” of $v$, i.e.,

\[\text{KINGDOM}(v) \overset{\text{def}}{=} \{v\} \cup \{w_{v,1}, w_{v,2}, \ldots, w_{v,2^i-1}\}\]

and $|\text{KINGDOM}(v)| = 2^i$.

If we can show that these kingdoms are disjoint for two different active nodes, then we are done.

**Proof by contradiction:** Suppose not. Suppose there are two nodes $u$ and $v$ such that $u \neq v$ and $\text{KINGDOM}(u) \cap \text{KINGDOM}(v) \neq \emptyset$ (after some round $i$, $1 \leq i \leq \log n$). Let $x$ be such that $x \in \text{KINGDOM}(u) \cap \text{KINGDOM}(v)$. Since an active node obviously cannot belong to the kingdom of another active node, this $x$ equals neither $u$ nor $v$, and therefore

\[x \in \{w_{u,1}, w_{u,2}, \ldots, w_{u,2^i-1}\} \cap \{w_{v,1}, w_{v,2}, \ldots, w_{v,2^i-1}\}\]

that is, both $u$ and $v$ have sent their respective probe-messages to $x$. Without loss of generality, let $ID_u > ID_v$.

- Case 1 ($u$ sends a probe-message to $x$ before $v$ does): Suppose $u$ sends a probe-message to $x$ at round $j$ and $v$ sends a probe-message to $x$ at round $j'$, where $1 \leq j < j' \leq \log n$. If $x$ had seen an ID higher than $ID_u$ up until round $j$, then $x$ immediately informs $u$ and $u$ becomes inactive. Contradiction. So suppose not. Then, after round $j$, $x$ sets its local variables $L_x$ and $N_x$ to $ID_u$ and $u$ respectively. Let $k > j$ be the smallest integer such that $x$ receives a probe-message from a neighbor

\[\text{...}\]
We will show a lower bound of $\Omega(n \log n)$ messages. Another very interesting question is whether explicit leader election (i.e., where all nodes know the identity of the leader) can be performed in $O(n)$ messages in diameter-two graphs (this is true for complete graphs, but not for diameter three and beyond). Also removing the assumption of the knowledge of $n$ (or showing that it is not possible) for deterministic algorithms with $O(n)$ message complexity and running in $O(1)$ rounds is open.

REFERENCES


We show that by constructing the Bordered Hessian matrix $H^B$ of the Lagrange function. Let $L_{ij}^* = \frac{\partial^2 \mathcal{L}}{\partial x_i \partial x_j} |_{X^*}$, where $\mathcal{L}$ is the Lagrange function as defined in Equation 4. Then

$$H^B = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & L^*_{11} & L^*_{12} & \cdots & L^*_{1n} \\
1 & L^*_{21} & L^*_{22} & \cdots & L^*_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & L^*_{n1} & L^*_{n2} & \cdots & L^*_{nn}
\end{bmatrix}$$

We note that $L^*_{ij} = \frac{2 \log \left( \frac{C}{n^2} \right) - 1}{\left( \frac{n^2}{C} \right)^2} - \lambda^*$ for all $1 \leq i \leq n$, and $L^*_{ij} = 0$ for all $(i,j)$ such that $i \neq j$. Hence

$$H^B = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & \frac{2 \log \left( \frac{C}{n^2} \right) - 1}{\left( \frac{n^2}{C} \right)^2} - \lambda^* & 0 & \cdots & 0 \\
1 & 0 & \frac{2 \log \left( \frac{C}{n^2} \right) - 1}{\left( \frac{n^2}{C} \right)^2} - \lambda^* & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & \frac{2 \log \left( \frac{C}{n^2} \right) - 1}{\left( \frac{n^2}{C} \right)^2} - \lambda^*
\end{bmatrix}$$

We show that $H^B$ is positive definite (which is a sufficient condition for $X^*$ to be a local minima) by checking the signs of the leading
principal minors. For any $1 \leq i \leq n$, $|H^R_i|$ is the determinant of a square matrix of dimension $i + 1$, and is given by

$$|H^R_i| = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 2 \log (C/n) - 1/(C/n)^3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 2 \log (C/n) - 1/(C/n)^3 - \lambda^* \end{vmatrix}$$

$$= -i(2 \log (C/n) - 1/(C/n)^3 - \lambda^*)^{i-1}.$$ 

But $2 \log (C/n) - 1/(C/n)^3 > 0$ 

[since $C \geq n\sqrt{2}, 2 \log (C/n) - 1 > 0$]

and $\lambda^* = -\log (C/n)/(C/n)^2 < 0$. Hence

$$2 \log (C/n) - 1/(C/n)^3 - \lambda^* > 0$$

$$\implies -i(2 \log (C/n) - 1/(C/n)^3 - \lambda^*)^{i-1} < 0$$

i.e., $|H^R_i| < 0$ for all $1 \leq i \leq n$ 

$\implies H^R$ is positive definite.

**Proof of Lemma 2.5.** We set $\delta = \frac{2 + \log n}{4 \log n}$. Then clearly $0 < \delta < 1$, and $1 - \delta = \frac{2 - \log n}{4 \log n}$. Again, from Lemma 2.3, we have that

$$\mu = E[X] > 2 + \frac{1}{2} \log n \implies (1 - \delta)\mu > (1 - \delta)(2 + \frac{1}{2} \log n)$$

$$= \frac{2}{4 + \log n}(2 + \frac{1}{2} \log n) = 1.$$

Then by Theorem 2.4, $Pr[X \leq 1] \leq Pr[X \leq (1 - \delta)\mu] \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}}\right)^{\mu}$. Now

$$\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} = \left(\frac{e^{2 + \log n}}{2 \log n}\right)^{\frac{1}{4 \log n}}$$

$$= \left(\frac{2 + \frac{1}{2} \log n)^2}{(2 + \frac{1}{2} \log n)^2}\right)^{\frac{1}{4 \log n}} < \left(\frac{2 + \frac{1}{2} \log n)^2}{ne^2}\right)^{\frac{1}{4 \log n}}$$

$$\implies \left(\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}}\right)^{\mu} < \left(\frac{2 + \frac{1}{2} \log n)^2}{ne^2}\right)^{\frac{1}{4 \log n}}$$

Hence, $Pr[X \leq 1] \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}}\right)^{\mu} < \frac{2 \frac{1}{2} \log n}{e^\delta n} < \frac{1}{e^\delta n}$, assuming $\frac{2 \frac{1}{2} \log n}{e^\delta n} < n^2$, which is asymptotically true.

**Proof of Lemma 5.1.** We consider the two cases — when $n$ is even and when $n$ is odd — separately in order to make the presentation simpler.

**Case 1:** $n = 2k$ for some integer $k \geq 2$.

**Proof by contradiction:** Suppose that there are three or more nodes that have had $\lceil \frac{n}{3} \rceil = k$ or more nodes removed each (either by themselves or by their neighbors). Let $u_v$, and $w$ be three such nodes. Since an edge is removed only if one of the incident nodes has a higher ID than the other, all of $(u, v)$, $(v, w)$, and $(w, u)$ cannot have been removed. Thus the total number of edges removed is at least $3k - 2 > 2k - 1$, which contradicts Observation 4.

**Case 1:** $n = 2k + 1$ for some integer $k \geq 1$.

**Proof by contradiction:** Suppose that there are three or more nodes that have had $\lceil \frac{n}{3} \rceil = k + 1$ or more edges removed each (either by themselves or by their neighbors). Let $u_v$, and $w$ be three such nodes. Since an edge is removed only if one of the incident nodes has a higher ID than the other, all of $(u, v)$, $(v, w)$, and $(w, u)$ cannot have been removed. Thus the total number of edges removed is at least $3(k + 1) - 2 > 2k$, which contradicts Observation 4.

**Proof of Lemma 5.2.** Clearly $G'$ is not of diameter one since the node with the smallest ID in $V$ always drops at least one edge.

Next we show that for any $u, v \in V$, either $u$ and $v$ are directly connected in $G'$ or $\exists w \in V$ such that $(u, w) \in E'$ and $(w, v) \in E'$.

We consider the two cases — when $n$ is even and when $n$ is odd — separately in order to make the presentation simpler.

**Case 1:** $n = 2k$ for some integer $k \geq 2$.

**Proof by contradiction:** Suppose that there are three or more nodes that have had $\lceil \frac{n}{3} \rceil = k$ or more nodes removed each (either by themselves or by their neighbors). Let $u_v$, and $w$ be three such nodes. Since an edge is removed only if one of the incident nodes has a higher ID than the other, all of $(u, v)$, $(v, w)$, and $(w, u)$ cannot have been removed. Thus the total number of edges removed is at least $3k - 2 > 2k - 1$, which contradicts Observation 4.

**Proof of Lemma 5.2.** Clearly $G'$ is not of diameter one since the node with the smallest ID in $V$ always drops at least one edge.
Since no node exists in $G'$ which has had $\lceil \frac{n}{2} \rceil = k$ or more edges removed, every node in $G'$ has degree at least $(n - 1) - (k - 1) = k$. Thus for any $u, v \in V$, if $(u, v) \not\in E'$, then there are at least $k + k - (n - 2) = 2$ nodes in $V \setminus \{u, v\}$ that are common neighbors to both $u$ and $v$.

**Case 1:** $n = 2k + 1$ for some integer $k \geq 1$.

Since no node exists in $G'$ which has had $\lceil \frac{n}{2} \rceil = k + 1$ or more edges removed, that implies that every node in $G'$ has degree at least $(n - 1) - k = k$. Thus for any $u, v \in V$, if $(u, v) \not\in E'$, then there is at least $k + k - (n - 2) = 1$ node in $V \setminus \{u, v\}$, which is a common neighbor to both $u$ and $v$.